

# Knot or Unknot ?

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## Contents

<b>A Natural Question</b>	<b>1</b>
<b>Unknot</b>	<b>2</b>
<b>Some Sample Calculations</b>	<b>3</b>
<b>References</b>	<b>4</b>
<b>Session Information</b>	<b>4</b>

## A Natural Question

Let  $A_n$  be the space of  $n$  or fewer disjoint arcs in the circle. In the special case  $n=0$ , we define  $A_0$  to be the 2 improper arcs: the empty arc and the full circle. Thus  $A_0$  has only 2 points, just like  $\mathbb{S}^0 = \{-1, 1\}$ . We have the inclusions:

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1} \subseteq \dots$$

We make  $A_n$  into a metric space as follows. Let  $V$  and  $W$  be collections of disjoint arcs in  $A_n$ , so  $V, W \subseteq \mathbb{S}^1$ . Let  $\mathbf{1}_V$  and  $\mathbf{1}_W$  be the indicator functions for  $V$  and  $W$ . Define the metric  $d(V, W) := \int_{\mathbb{S}^1} |\mathbf{1}_V - \mathbf{1}_W| d\theta$ . So  $A_n$  can be regarded as a subset of  $L^1(\mathbb{S}^1)$  and it inherits a metric from that. The metric can be viewed in another way; the symmetric difference of  $V$  and  $W$  is also a collection of arcs, and  $d(V, W)$  is the total length of the arcs in this symmetric difference. The corresponding functions in the package are `arcsdistance()` and `arcssymdiff()`.

In the **User Guide** vignette it is shown that there are homeomorphisms

$$A_n \longleftrightarrow \partial Z_n \longleftrightarrow \mathbb{S}^{2n} \tag{1}$$

where  $Z_n$  is the polar zonoid in  $\mathbb{R}^{2n+1}$ . These inclusions and homeomorphisms induce embeddings  $\mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2}$  in this large commutative diagram.

$$\begin{array}{cccccccccccc}
A_0 & \subseteq & A_1 & \subseteq & A_2 & \subseteq & \cdots & \subseteq & A_n & \subseteq & A_{n+1} & \subseteq & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & & \\
\partial Z_0 & \hookrightarrow & \partial Z_1 & \hookrightarrow & \partial Z_2 & \hookrightarrow & \cdots & \hookrightarrow & \partial Z_n & \hookrightarrow & \partial Z_{n+1} & \hookrightarrow & \cdots \\
\updownarrow & & \updownarrow & & \updownarrow & & & & \updownarrow & & \updownarrow & & \\
\mathbb{S}^0 & \hookrightarrow & \mathbb{S}^2 & \hookrightarrow & \mathbb{S}^4 & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{S}^{2n} & \hookrightarrow & \mathbb{S}^{2n+2} & \hookrightarrow & \cdots
\end{array}$$

The embeddings in the middle row are induced from the inclusions on the top row. The middle row is almost surely a Whitney stratification, see [1], but I have not checked this. The embeddings in the bottom row are induce from those in the middle row. We will soon see that the embedding

$$\mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2} \tag{2}$$

is *not* the standard one, which is formed by appending two zeros. The codimension is 2, which is the only codimension where knotting of spheres occurs (at least in the PL category, see [2]). So it is natural to ask:

Q: Is the embedding  $\mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2}$  unknotted ?

i.e. is it isotopic to the standard embedding  $\mathbb{S}^{2n} \subseteq \mathbb{S}^{2n+2}$  ?

## Unknot

In this section we show that the embedding (2) is *not* knotted. For the purposes of this vignette, we use the standard order for the basis of the trigonometric polynomials:

$$1, \cos(\theta), \sin(\theta), \cos(2\theta), \sin(2\theta), \dots, \cos(n\theta), \sin(n\theta) \quad \theta \in [0, 2\pi] \tag{3}$$

and put 1 at the beginning instead of the end, as in the **polarzonoid** package.

This embedding is the composition:

$$\mathbb{S}^{2n} \rightarrow \partial Z_n \hookrightarrow \partial Z_{n+1} \rightarrow \mathbb{S}^{2n+2} \tag{4}$$

Focus first on the middle embedding, which is the composition:

$$\partial Z_n \rightarrow A_n \subseteq A_{n+1} \rightarrow \partial Z_{n+1} \tag{5}$$

If a point  $p \in \partial Z_n$  maps to a set of disjoint arcs  $a \in A_n$ , then this composition (5) is:

$$p \mapsto \left( p, \int_a \cos((n+1)\theta) d\theta, \int_a \sin((n+1)\theta) d\theta \right) \tag{6}$$

And since  $a$  is a function of  $p$ , () can be written:

$$p \mapsto ( p, v(p) ) \quad \text{for a function } v : \partial Z_n \rightarrow \mathbb{R}^2 \tag{7}$$

Returning now to (4), it is convenient to translate  $\partial Z_n$  and  $\partial Z_{n+1}$  so their centers are at 0. The center of  $\partial Z_n$  is  $(\pi, 0, \dots, 0)$  so only the first coordinate is changed, and (7) is still valid. Denoting the centered sets by adding a prime ', the composition is now

$$\mathbb{S}^{2n} \rightarrow \partial Z'_n \hookrightarrow \partial Z'_{n+1} \rightarrow \mathbb{S}^{2n+2} \tag{8}$$

After these translations, the first and last maps are simple multiplication and division by positive real functions, and (8) is:

$$u \mapsto (\alpha(u)u, v(\alpha(u)u)) / (\alpha^2(u) + |v(\alpha(u)u)|^2)^{1/2} \quad \text{where } |u| = 1 \text{ and } \alpha(u) > 0 \quad (9)$$

This can be simplified to

$$u \mapsto (\beta(u)u, w(u)) \quad \text{where } \beta(u) > 0 \text{ and } w(u) \in \mathbb{R}^2 \text{ and } |w(u)| < 1 \quad (10)$$

We are done if we can show:

**Theorem:** Any embedding  $f : \mathbb{S}^{2n} \hookrightarrow \mathbb{S}^{2n+2}$  that has the form

$$f(u) = (\beta(u)u, w(u)) \quad \text{where } \beta(u) > 0 \text{ and } w(u) \in \mathbb{R}^2 \text{ and } |w(u)| < 1 \quad (11)$$

is isotopic to the standard embedding.

**Proof:** First note that  $\beta(u)$  and  $w(u)$  are not independent of each other. In fact we have:

$$1 = |f(u)|^2 = \beta(u)^2 \cdot 1^2 + |w(u)|^2 \quad (12)$$

and so  $\beta(u) = (1 - |w(u)|^2)^{1/2}$  and  $f$  depends only on  $w : \mathbb{S}^{2n+2} \rightarrow \mathbb{R}^2$ . Now  $\mathbb{R}^2$  is contractible so  $w$  is homotopic to the 0 map. In fact, if we let the homotopy parameter be  $t \in [0, 1]$ , we can define  $w_t(u) := (1-t)w(u)$ . Note that  $w_1$  is the 0 map, and we always have  $|w_t(u)| < 1$ . So each intermediate  $w_t$  defines an intermediate  $f_t$ , which is an embedding. And the final  $f_1$  is the standard embedding:  $f_1(u) = (u, 0, 0)$ .  $\square$

## Some Sample Calculations

```
library( polarzonoid )
```

In this section, we verify that the induced embedding  $\mathbb{S}^6 \hookrightarrow \mathbb{S}^8$  has the correct form (10), for a few test cases. Since the software functions put the constant term last (instead of first) the indexes for  $\mathbb{S}^6$  are 1,2,3,4,5,6,9 (omitting 7 and 8).

```
idx = c(1:6,9)

# make a random unit vector in S^6
set.seed(0)
u = rnorm(7) ; u = u / sqrt( sum(u^2) )
# embed into S^8
up = spherefromarcs( arcsfromsphere(u), n=4 )
beta = up[idx] / u ; beta
```

```
## [1] 0.9621536 0.9621536 0.9621536 0.9621536 0.9621536 0.9621536 0.9621536
```

```
range( diff(beta) )
```

```
## [1] -5.295764e-14 8.859580e-14
```

So all the relevant coordinates are scaled by the same number, up to numerical truncation.

Now repeat this test many times.

```

count = 50
umat = array( rnorm(count*7), dim=c(count,7) )
umat = umat / sqrt( rowSums(umat^2) )
upmat = t( apply( umat, 1, function(u) { spherefromarcs( arcsfromsphere(u), n=4 ) } ) )
betamat = upmat[ ,idx] / umat
delta = apply( betamat, 1, diff )
range( delta )

```

```
## [1] -7.925771e-11  3.887302e-11
```

Once again, the relevant coordinates are scaled by the same number, up to numerical truncation.

## References

- [1] WIKIPEDIA CONTRIBUTORS. *Whitney conditions* — *Wikipedia, the free encyclopedia* [online]. 2022. Available at: [https://en.wikipedia.org/w/index.php?title=Whitney\\_conditions&oldid=1119503349](https://en.wikipedia.org/w/index.php?title=Whitney_conditions&oldid=1119503349). [Online; accessed 3-June-2025]
- [2] ZEEMAN, E. C. Unknotting combinatorial balls. *Annals of Mathematics*. 1963, **78**(3), 501–526.

## Session Information

This document was prepared Mon Aug 25, 2025 with the following configuration:

```

R version 4.5.1 (2025-06-13 ucrt)
Platform: x86_64-w64-mingw32/x64
Running under: Windows 11 x64 (build 26100)

```

```

Matrix products: default
  LAPACK version 3.12.1

```

```

locale:
[1] LC_COLLATE=C
[2] LC_CTYPE=English_United States.utf8
[3] LC_MONETARY=English_United States.utf8
[4] LC_NUMERIC=C
[5] LC_TIME=English_United States.utf8

```

```

time zone: America/Los_Angeles
tzcode source: internal

```

```

attached base packages:
[1] stats      graphics  grDevices  utils      datasets  methods   base

```

```

other attached packages:
[1] zonohedra_0.5-0  rgl_1.3.18      flextable_0.9.7  polarzonoid_0.2-0

```

```

loaded via a namespace (and not attached):

```

[1] katex_1.5.0	jsonlite_2.0.0	qpdf_1.3.5
[4] compiler_4.5.1	pdftools_3.5.0	equatags_0.2.1
[7] tinytex_0.57	Rcpp_1.0.14	zip_2.3.3
[10] xml2_1.3.8	av_0.9.4	magick_2.8.6
[13] jquerylib_0.1.4	fontquiver_0.2.1	systemfonts_1.2.3
[16] textshaping_1.0.1	uuid_1.2-1	yaml_2.3.10
[19] fastmap_1.2.0	R6_2.6.1	microbenchmark_1.5.0
[22] gdtools_0.4.2	curl_6.2.2	knitr_1.50
[25] htmlwidgets_1.6.4	logger_0.4.0	openssl_2.3.2
[28] bslib_0.9.0	rlang_1.1.6	V8_6.0.3
[31] cachem_1.1.0	xfun_0.52	sass_0.4.10
[34] cli_3.6.5	magrittr_2.0.3	digest_0.6.37
[37] grid_4.5.1	base64enc_0.1-3	askpass_1.2.1
[40] lifecycle_1.0.4	evaluate_1.0.3	glue_1.8.0
[43] data.table_1.17.2	fontLiberation_0.1.0	officer_0.6.8
[46] ragg_1.4.0	xslt_1.5.1	fontBitstreamVera_0.1.1
[49] rmarkdown_2.29	tools_4.5.1	htmltools_0.5.8.1